Gauss-Bonnet Thm:

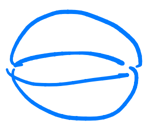
(Goal) Let M be a regular orientable compact surface with $\partial M = \emptyset$.

Then $\displaystyle\int_M K\,dA = 2\pi\,\underline{\chi(M)} = 4\pi(1-g)$

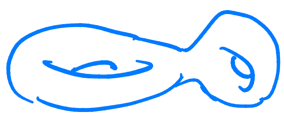
euler characteristic of M.

where g = genus of M.

$g = 1 \Rightarrow \int_M K\,dA = 0$

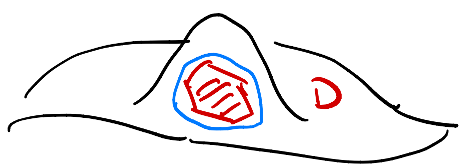


$g = 0 \Rightarrow \int_M K\,dA = 4\pi$



$g = 2 \Rightarrow \int_M K\,dA = -4\pi$, etc....

Step 1: local version (??).



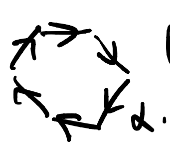
Consider $D \subseteq M$ with "singular" boundary. and D small enough s.t. D is covered by local parametrization.

i.e. $X : U \to M$ s.t. $X(U) \supsetneq D$.

Further simplification: might assume $X : U \to M$ is isothermal.

i.e. $[g] = \begin{bmatrix} e^{2f} & 0 \\ 0 & e^{2f} \end{bmatrix}$ for some $f : U \to \mathbb{R}$.

Q: what should be the local Gauss-Bonnet thm??

Set-up: $D =$  (e.g.) in M.

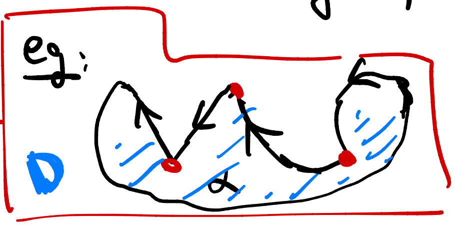
Defn: $\alpha : [a,b] \to M$ is a simple closed curve which is piece-wise differentiable and regular

① closed : $\alpha(a) = \alpha(b)$

② simple : no self intersection (i.e. $\forall\, t \neq s \in [a,b)$, $\alpha(t) \neq \alpha(s)$)

③ piecewise differentiable (regular) : $\exists\ a = t_0 < t_1 < \cdots < t_{k+1} = b$ s.t. $\alpha\big|_{(t_i, t_{i+1})}$ is differentiable (regular).

eg:



like this.

Assume $\alpha_i = \alpha\big|_{(t_i, t_{i+1})}$ are all positively oriented

i.e. its normal n satisfies :

$N = \alpha' \times n$ and n points toward $\mathrm{int}(D)$.

Now compute $\int_D K\, dA$: (assume $D \subseteq X(U)$ in Isothermal coordinate)

Using local formula of K in term of Γ_{ij}^k and $g_{ij} = e^{2f}\delta_{ij}$.

$\Rightarrow\ K = -e^{-2f}\Delta_{\mathbb{R}^2} f$ as a fcn on U.

Hence $\int_D K\, dA = \int_{X^{-1}(D)} (-e^{-2f}\Delta f)\cdot e^{2f}\, du\, dv$

$= \int_{X^{-1}(D)} \underline{\underline{-\Delta f}}\, .\ du\, dv$ (as a double integral in $\mathbb{R}^2$)

Green thm for domain with piecewise smooth boundary.

$= -\int_\gamma \langle \nabla f, \nu \rangle\, d\tau$ where $X(\gamma) = \alpha$.

arc-length parametrization as curve in $\mathbb{R}^2$.

Q: what is geometric meaning of $\int_\gamma \langle \nabla f, \nu \rangle \, d\tau$??

In $X : \mathcal{U} \to M$, $\begin{cases} e_1 = \dfrac{X_u}{\|X_u\|} = e^{-f} X_u \\ e_2 = \dfrac{X_v}{\|X_v\|} = e^{-f} X_v \end{cases}$ are O.N.

and $N = e_1 \times e_2$ (by assumption)

In the regular part of $\alpha : [a,b] \to M$ (parametrized by arc-length)

$$\alpha' = \cos\theta \cdot e_1 + \sin\theta \, e_2 \quad \text{for some smooth } \theta.$$

$$\Rightarrow \quad n = -\sin\theta \, e_1 + \cos\theta \, e_2$$

$$\Rightarrow k_g = \langle \alpha'', n \rangle$$

$$= \theta' + \langle \cos\theta \, e_1' + \sin\theta \, e_2', -\sin\theta \, e_1 + \cos\theta \, e_2 \rangle$$

$$= \theta' + \cos^2\theta \langle e_1', e_2 \rangle - \sin^2\theta \langle e_1, e_2' \rangle$$

$$= \theta' + \langle e_1', e_2 \rangle \quad \text{where}$$

$$\langle e_1', e_2 \rangle = \langle (e^{-f} X_u)', (e^{-f} X_v) \rangle$$

$$= e^{-2f} \langle X_u', X_v \rangle = \langle X_{uu} u' + X_{uv} v', X_v \rangle e^{-2f}.$$

- $\langle X_{uu}, X_v \rangle = -\langle X_u, X_{uv} \rangle \qquad (\because g_{uv} = 0)$

$$= -\frac{1}{2}\langle X_u, X_u \rangle_v$$

$$= -\frac{1}{2}\left(e^{2f}\right)_v = -e^{2f} \cdot f_v.$$

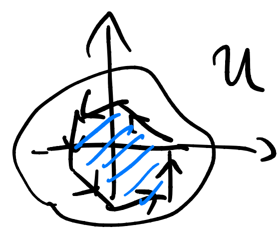
- $\langle X_{uv}, X_v \rangle = \frac{1}{2}\langle X_v, X_v \rangle_u = \frac{1}{2}\left(e^{2f}\right)_u = e^{2f} f_u.$

$\therefore \quad \kappa_g = \theta' + \underbrace{\left(-f_v u' + f_u v'\right)}$ on regular part of α.

(Here derivative $= \frac{d}{ds}$) (i.e. regular part of γ)

On $\gamma : [a,b] \to \mathcal{U}$ ($s \in$)

where $X(\gamma) = \alpha$.



$$\gamma(s) = (u(s), v(s)), \quad s \in [a,b]$$

$$= \gamma(s(\tau)) \quad \text{where } \tau = \text{arc-length parameter of } \gamma \text{ wrt std metric on } \mathbb{R}^2.$$

Then $\langle \nabla f, \nu \rangle$

$$= \langle (f_u, f_v), (v_\tau, -u_\tau) \rangle$$

$$= -u_\tau \cdot f_v + v_\tau f_u = \left(-f_v u' + f_u v'\right) \frac{ds}{d\tau}$$

$$\therefore \int_{\mathcal{U}} K \, dA = \int_\gamma -\langle \nabla f, \nu \rangle \, d\tau.$$

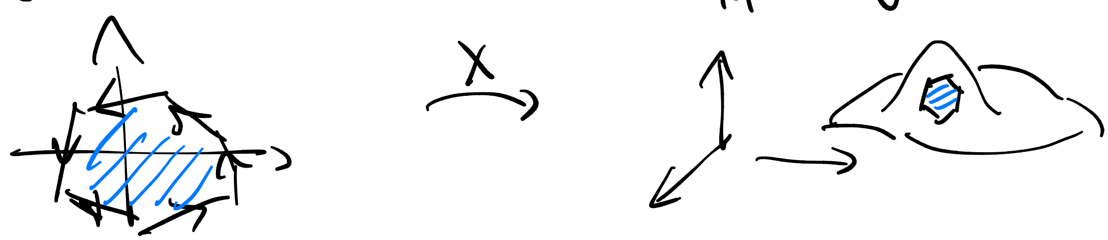
$$= \int_a^b \left(f_v u' - f_u v'\right) ds$$

$$= \int_a^b -k_g \, ds + \int_a^b \frac{d\theta}{ds} \, ds$$

$$= \int_a^b -k_g \, ds + \sum_{i=0}^{k} \underbrace{\int_{t_i}^{t_{i+1}} \frac{d\theta}{ds} \, ds}$$

$$= \sum_{i=0}^{k} \lim_{t \to t_{i+1}^-} \theta(t) - \lim_{t \to t_i^+} \theta(t) = ??$$

<u>Recall</u> $\cos \theta(s) = \langle \alpha'(s), e_1(s) \rangle_M$ along α



θ = angle between $\alpha'(s)$ and $e_1(s)$.

translate to picture on γ :

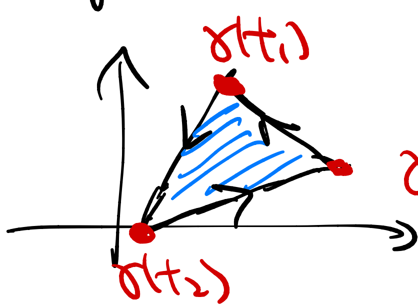
$$\alpha' = \cos\theta \cdot e_1 + \sin\theta \, e_2$$
$$= u' X_u + v' X_v$$
$$= u' e^f e_1 + v' e^f e_2$$

$$\Rightarrow \gamma'(s) = e^{-f} (\cos\theta, \sin\theta)$$

i.e. θ = angle between γ' and $(1,0)$.

wrt Euclidean metric

Eg 1)



$\gamma(t_0) = \gamma(t_3)$ ← simple closed curve.

$$\Rightarrow \sum_{i=0}^{k} \int_{t_i}^{t_{i+1}} \frac{d\theta}{ds}\, ds = \sum_{i=0}^{k} 0 = 0$$

$\because \gamma|_{[t_i, t_{i+1}]} =$ linear.

Interpret this as follows:

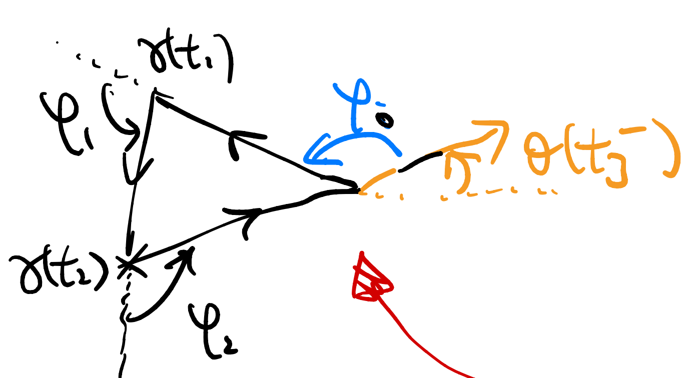
Denote
$$\begin{cases} \theta(t_i^+) = \lim_{t \to t_i^+} \theta\left(\gamma|_{(t_i, t_{i+1})}, t\right) \\ \theta(t_i^-) = \lim_{t \to t_i^-} \theta\left(\gamma|_{(t_{i-1}, t_i)}, t\right) \end{cases}$$

then
$$\sum_{i=0}^{k} \int_{t_i}^{t_{i+1}} \frac{d\theta}{ds}\, ds = \sum_{i=0}^{k} \theta(t_{i+1}^-) - \theta(t_i^+).$$

$$\begin{aligned} = \quad & \theta(t_{k+1}^-) - \theta(t_0^+) \\ & + \theta(t_k^-) - \theta(t_k^+) \\ & + \cdots \\ & + \theta(t_1^-) - \theta(t_1^+). \end{aligned}$$

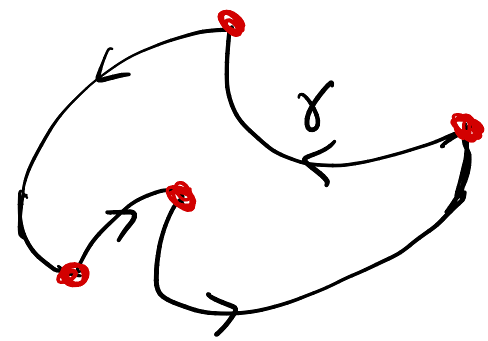
$$= \begin{aligned} & \theta(t_3^-) - \theta(t_0^+) \\ & + \theta(t_2^-) - \theta(t_2^+) \\ & + \theta(t_1^-) - \theta(t_1^+) \end{aligned}$$

$$= 2\pi - \sum_{i=0}^{k} \varphi_i$$



where φ_i : exterior angle at $\gamma(t_i)$

More generally we allow $\gamma|_{[t_i, t_{i+1}]}$ to be differentiable curve.



(eg).

Thm (From topology, the above is true in general)

Let $\gamma : [a,b] \to \mathbb{R}^2$ be piecewise regular, simple closed curve with $\gamma(a) = \gamma(b)$.

Let $a = t_0 < t_1 < \cdots < t_{k+1} = b$ be s.t.

$\gamma|_{(t_i, t_{i+1})}$ = regular curve, parametrized by arc-length

s.t. $\gamma' = (\cos\theta_i, \sin\theta_i)$ on $[t_i, t_{i+1}]$.

And define φ_i be exterior angle defined to be angle at $\gamma(t_i)$ from $\gamma'(t_i^-)$ to $\gamma'(t_i^+)$.

Then $$\sum_{i=1}^{k} \left[\theta(t_{i+1}^-) - \theta(t_i^+)\right] + \sum_{i=1}^{k} \varphi_i = \pm 2\pi.$$

depending on orientation of γ.

$\boxed{\text{* } 2\pi \text{ comes from the loop!!}}$

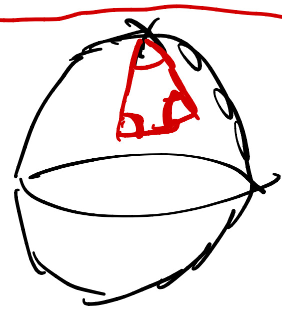
$\Downarrow$

Local Gauss–Bonnet thm

Let $X : U \to M$ be an orthonormal parametrization which is orientation preserving. Let $\alpha = X(\gamma)$ be a simple closed curve on M, for some simple closed curve which is piecewise regular, then for $D = X(R)$ where R = region bdd by γ (exists by Jordan curve thm),

$$\int_D K\,dA + \int_\alpha k_g\,ds + \sum_{i=0}^{k} \varphi_i = 2\pi.$$

Eg:

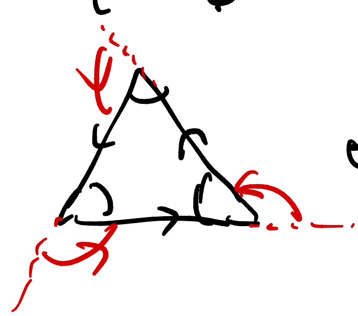


$M = S^2$ = earth

$D = \Delta$ formed by three geodesic.

then
$$\begin{cases} k_g = 0 \text{ on } \gamma \\ K_g = 1 \text{ on } D. \\ \int_D dA = 2\pi - \sum_{i=1}^{3} \varphi_i > 0 \end{cases}$$

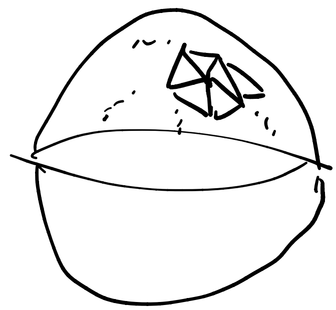
v.s.



on plane $\cong \mathbb{R}^2$.

$$2\pi - \sum_{i=1}^{3} \varphi_i = 0.$$

global version (cpt w/o boundary)



Cover M by small triangle:

- **Defn**: A triangle (topologically) in M is a closed subset T in form of $\phi(T')$ where T' is triangle in $\mathbb{R}^2$, ϕ is homeomorphism.

  Vertex, edge and faces are defined to be image of that of $T' \subseteq \mathbb{R}^2$.

- Any cpt surface w/o boundary admits a triangulation

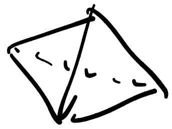
  i.e. $\exists\ T_1, T_2, \cdots, T_n$ triangle s.t.

  $T_i \cap T_j$ is either empty, single vertex

  OR single edge in common $\forall i \neq j$

$\chi(M)$, Euler characteristic of M

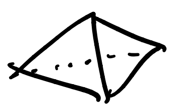
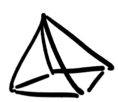
$$= V - E + F \quad \begin{cases} V = \text{total no. of vertex} \\ E = \text{total no. of edge} \\ F = \text{total no. of faces} \end{cases}$$

eg:  $(\cong S^2)$ $\begin{cases} F = 4 \\ E = 6 \\ V = 4 \end{cases}$

$$\Rightarrow \chi(M) = 4 - 6 + 4 = 2$$

Fact from topology:

$\boxed{\text{Thm: } \chi(M) \text{ is independent of choice of triangulation (topological quantity)}}$

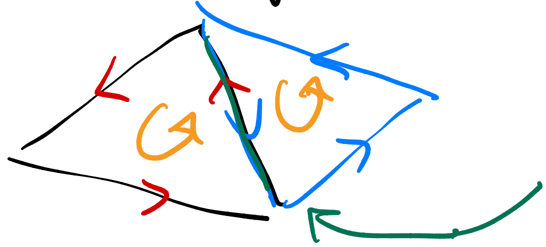
 $\cong$  $\cong S^2$

$F = 5, \; E = 8, \; V = 5$

$$\Rightarrow \chi(M) = 2$$

- <u>Thm</u> (in topology) $\chi(M) = 2 - 2g$ (g = genus of M.)

- Might refine triangulation s.t. each triangle is inside isothermal coord.



the edge has opposite orientation.

Proof of GB thm:

Let $\{T_i\}_{i=1}^{N}$ be a triangulation of M s.t. each T_i is inside an isothermal coordinate.

$$\Rightarrow \int_{\partial T_j} k_g \, ds + \int_{D_j} K \, dA + \sum_{i=1}^{3} \varphi_{ij} = 2\pi$$

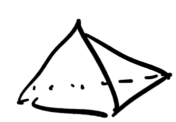
$$\underbrace{\sum_{j=1}^{N} \int_{\partial T_j} k_g \, ds}_{=\,0} + \underbrace{\sum_{j=1}^{N} \int_{D_j} K \, dA}_{=\,\int_M K\, dA} + \sum_{j=1}^{N} \sum_{i=1}^{3} \varphi_{ij} = 2\pi N.$$

0 ($\because$ M = cpt w/o boundary)

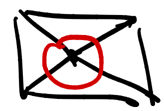
$\int_M K \, dA$

Final Claim: $2\pi N - \sum_{j=1}^{N} \sum_{i=1}^{3} \varphi_{ij} = 2\pi \chi(M).$

Pf: $\because$ M = cpt w/o boundary

$\therefore \begin{cases} 3F = 2E \\ N = F \end{cases}$ (eg. )

Denote $\pi - \varphi_{ij} = \theta_{ij}$ to be interior angle then $\sum \theta_{ij} = 2\pi \cdot V$



$$\therefore 2\pi N - \sum_{ij} \varphi_{ij} = \sum \theta_{ij} - \pi F$$

$$= 2\pi V - \pi F$$

$$= 2\pi V + 2\pi F - 3\pi F$$

$$= 2\pi (V + F - E) = 2\pi \cdot \chi(M)$$ #

END of Courses !! :)